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Interpolation in Minimum Semi-Norm and Multivariate B-Splines

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1. INTRODUCTION

In [4] de Boor defined the notion of multivariate B-spline and since then it has been widely studied by a number of authors, e.g., [3, 5, 7]. Much has been discovered concerning its approximation properties but as yet little is known concerning interpolation. In the one-dimensional case, interpolating spline functions are found as interpolating functions of minimum semi-norm, the so-called natural splines (see [12]). Interpolating functions of minimum semi-norm have been studied widely and in much generality, e.g., see [9]. In this paper we consider such problems related to multivariate B-splines. The resulting interpolating functions will be integrals of linear combinations of multivariate B-splines, but will not in general be such linear combinations themselves.

In Section 3 we consider interpolation by functions of minimum semi-norm in a setting general enough to cover all cases later considered. This setting requires a certain space of interpolating functions, which in our case will always be multivariate polynomials. Such polynomial interpolation, for different types of interpolation, is considered in Section 2. This also leads to a simple proof of a result of Dahmen and Micchelli [3] on the dimension of spaces spanned by multivariate B-splines. Finally, Section 4 considers particular cases of Section 3 related to multivariate B-splines and examines the interpolating functions.

2. INTERPOLATION BY POLYNOMIALS

In [8] Kergin introduced a new form of interpolation by multivariate polynomials. A different form of multivariate polynomial interpolation was given by Hakopian [5]. Here we give a more general form of interpolation which includes both of the above.

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We shall be working throughout this paper in s-dimensional Euclidean space R^s, $s \geqslant 1$. For $j = 1,\dots, s$, D_j denotes $\partial/\partial x_j$ and for y in R^s, D_y denotes the directional derivative $\sum_{j=1}^s y_j D_j$. For any multi-index $\alpha = (\alpha_1,\dots, \alpha_s)$, D^α denotes $D_1^{\alpha_1} \cdots D_s^{\alpha_s}$ and $|\alpha| := \alpha_1 + \cdots + \alpha_s$. The cardinality of a finite set A (counting multiplicities) is denoted by $|A|$, and P_m denotes the space of polynomials in R^s of (total) degree at most m.

Now for f in $C(R^s)$ and points $x^0,\dots, x^n$ in R^s (possibly coincident), we follow [10] in writing

$$\int_{[x^0,\dots,x^n]} f := \int_{S^n} f(t_0 x^0 + \cdots + t_n x^n)\, dt_1 \cdots dt_n,$$

where $S^n := \{(t_1,\dots, t_n)\colon t_j \geqslant 0, j = 1,\dots, n,\ t_1 + \cdots + t_n \leqslant 1\}$ is the standard n-simplex and

$$t_0 = 1 - (t_1 + \cdots + t_n).$$

We also recall, as in [10], that for $s = 1$ we have the following, called the Hermite–Gennochi formula:

$$\int_{[t_0,\dots,t_n]} f^{(n)} = [t_0,\dots, t_n] f, \tag{1}$$

where the right-hand side denotes the usual divided difference.

THEOREM 1. *Take points $x^0,\dots, x^n$ in R^s (possibly coincident) and an integer k with $0 \leqslant k \leqslant n$. Then for any function f in $C^{n-k}(R^s)$ there is a unique polynomial p in P_{n-k} such that for any $A \subset \{x^0,\dots, x^n\}$ with $|A| \geqslant k + 1$ and any multi-index α with $|\alpha| = |A| - k - 1$,*

$$\int_{[A]} D^\alpha p = \int_{[A]} D^\alpha f. \tag{2}$$

Moreover, we have

$$\begin{aligned} \frac{1}{k!} p(x) = {} & \int_{[x^0,\dots,x^k]} f + \sum_{j=0}^{k} \int_{[x^0,\dots,x^{k+1}]} D_{x-x^j} f \\ & + \sum_{0 \leqslant j_1 < j_2 \leqslant k+1} \int_{[x^0,\dots,x^{k+2}]} D_{x-x^{j_1}} D_{x-x^{j_2}} f + \cdots \\ & + \sum_{0 \leqslant j_1 < \cdots < j_{n-k} \leqslant n-1} \int_{[x^0,\dots,x^n]} D_{x-x^{j_1}} \cdots D_{x-x^{j_{n-k}}} f. \end{aligned} \tag{3}$$

Proof. We first show that p given by (3) satisfies condition (2). This was proved for $k=0$ by Micchelli and Milman in [11] and we follow the same approach in noting that it is sufficient to prove it for functions of the form $f(x)=g(\lambda_1 x_1+\cdots+\lambda_s x_s)=: g(\lambda\cdot x)$ for λ in R^s and g in $C^{n-k}(R)$.

Choose h in $C^n(R)$ with $h^{(k)}=g$. Let q denote the polynomial on R of degree at most n which interpolates h at the points $\lambda\cdot x^0,\ldots,\lambda\cdot x^n$, and put $\hat{p}(x)=q^{(k)}(\lambda\cdot x)$. Now take any set $A=\{y^0,\ldots,y^r\}\subset\{x^0,\ldots,x^n\}$, $r\geqslant k$, and α with $|\alpha|=r-k$. Then

$$\begin{aligned}
\int_{[A]} D^\alpha f &= \int_{[\lambda\cdot y^0,\ldots,\lambda\cdot y^r]} \lambda^\alpha g^{(r-k)} \\
&= \lambda^\alpha \int_{[\lambda\cdot y^0,\ldots,\lambda\cdot y^r]} h^{(r)} \\
&= \lambda^\alpha [\lambda\cdot y^0,\ldots,\lambda\cdot y^r] h \qquad &&\text{(by (1))} \\
&= \lambda^\alpha [\lambda\cdot y^0,\ldots,\lambda\cdot y^r] q \\
&= \lambda_\alpha \int_{[\lambda\cdot y^0,\ldots,\lambda\cdot y^r]} q^{(r)} \qquad &&\text{(by (1))} \\
&= \int_{[A]} D^\alpha \hat{p}.
\end{aligned}$$

Thus $\hat{p}$ satisfies condition (2), and it remains to show that $\hat{p}=p$ given by (3). Writing Newton's form for q, and recalling (1), gives

$$\begin{aligned}
q(t) &= h(\lambda\cdot x^0)+(t-\lambda\cdot x^0)\int_{[\lambda\cdot x^0,\lambda\cdot x^1]} h' + \cdots \\
&\quad + (t-\lambda\cdot x^0)\cdots(t-\lambda\cdot x^{n-1})\int_{[\lambda\cdot x^0,\ldots,\lambda\cdot x^n]} h^{(n)}.
\end{aligned}$$

Differentiating k times, putting $t=\lambda\cdot x$ and recalling $\hat{p}(x)=q^{(k)}(\lambda\cdot x)$, $f(x)=h^{(k)}(\lambda\cdot x)$, we see $\hat{p}=p$.

Finally, we show that the interpolating polynomial is unique. We must show that if p in P_{n-k} satisfies

$$\int_{[A]} D^\alpha p = 0 \tag{4}$$

for any $A\subset\{x^0,\ldots,x^n\}$, $|A|\geqslant k+1$, and $|\alpha|=|A|-k-1$, then $p=0$.

We prove this by induction on n, it being clearly true for $n=k$. Suppose it is true for $n=N\geqslant k$. Suppose p in P_{N+1-k} satisfies (4) for all

$A \subset \{x^0,..., x^{N+1}\}$, $|A| \geqslant k+1$, $|\alpha| = |A| - k - 1$. Then for any α with $|\alpha| = N + 1 - k$, we have

$$0 = \int_{[x^0,\ldots,x^{N+1}]} D^\alpha p = D^\alpha p/(N+1)!.$$

Thus $p \in P_{N-k}$. Since p satisfies (4) for any $A \subset \{x^0,..., x^N\}$, $|A| \geqslant k+1$, $|\alpha| = |A| - k - 1$, our induction hypothesis shows $p = 0$. ∎

We now list some comments on Theorem 2. It will be useful to first make a definition. For any set $A \subset R^s$, $[A]$ will denote its closed convex hull and $\mathrm{vol}_s A$ will denote its s-dimensional Lebesque measure. Then for any integer q, we say the points $x^0,..., x^n$ in R^s are *q-admissible* if for any set $A \subset \{x^0,..., x^n\}$ with $|A| = q + 1$, we have $\mathrm{vol}_s[A] > 0$. (In the terminology of [3], this says $P_{n+1,q+1}$ is admissible, where $P = \{x^0,..., x^n\}$).

Remark 1. For $k = 0$, the unique interpolation of Theorem 2 was proved by Kergin [8] and formula (3) was derived by Micchelli and Milman [11] who refer to such interpolation as Kergin interpolation. In this case we have in particular that p interpolates f at $x^0,..., x^n$. Moreover, when $x^0 = \cdots = x^n$, (3) reduces to the usual formula for the Taylor polynomial.

Remark 2. For $k = 1$, Theorem 2 was proved by Cavaretta *et al.* [1].

Remark 3. For $k = s - 1$, the unique interpolation was given by Hakopian [5] for s-admissible points, and for general points for $s = 2$. In [3] Dahmen and Micchelli (independently of the author) pointed out that Hakopian's interpolation could be derived by the method of our proof of Theorem 2.

Remark 4. In general the linear functionals given by (2) will not be linearly independent. Indeed it is shown in [10] that if $y = \sum_{j=0}^n \mu_j x^j$, $\sum_{j=0}^n \mu_j = 0$, then

$$\int_{[x^0,\ldots,x^n]} D_y f = -\sum_{j=0}^{n} \mu_j \int_{[x^0,\ldots,x^{j-1},x^{j+1},\ldots,x^n]} f. \tag{5}$$

For $k \leqslant s - 1$, repeated application of (5) can be used to reduce the linear functionals in (2) to a linearly independent collection, but in general an explicit description of this collection is very complicated. Such a description for $s = 2$, $k = 1$ is given in [5]. Such reduction may also allow us to weaken our smoothness criteria. To be precise, suppose $x^0,..., x^n$ are q-admissible for some q, $k + 1 \leqslant q \leqslant n$. Then by repeated application of (5), any term of the form

$$\int_{[A]} D^\alpha f, \qquad A \subset \{x^0,..., x^n\}, \quad |A| \geqslant q, \quad |\alpha| = |A| - k - 1, \tag{6}$$

can be expressed as a linear combination of terms of form (6) for $|A| = q$. Thus the interpolation of Theorem 1 can be extended to functions in $C^{q-k-1}(R^s)$, or indeed in $C^{q-k-1}([x^0, ..., x^n])$.

Remark 5. Finally, we note that for $k \geqslant s$, Theorem 1 gives us information about the dimension of certain spaces spanned by multivariate B-splines. For $x^0, ..., x^n$ in R^s with $\mathrm{vol}_s[x^0, ..., x^n] > 0$, the B-spline $M(\cdot \mid x^0, ..., x^n)$ was first defined in [4], and in [10] was shown to satisfy

$$\int_{R^s} M(x \mid x^0, ..., x^n) f(x)\, dx = n! \int_{[x^0, \ldots, x^n]} f \tag{7}$$

for any f in $C(R^s)$. We follow Dahmen and Micchelli [3] in using (7) as the definition of the B-spline. As pointed out by Höllig [7], we can also use (7) to define $M(\cdot \mid x^0, ..., x^n)$ for arbitrary $x^0, ..., x^n$ as a positive measure supported on $[x^0, ..., x^n]$. Then statements about the linear independence of linear functionals in (2) translate directly into statements about the linear independence of the corresponding B-splines. In particular we have

COROLLARY 1. *For $s - 1 \leqslant k \leqslant n$, suppose $x^0, ..., x^n$ are $(k+1)$-admissible points in R^s (possibly coincident). Then the space spanned by $\{M(\cdot \mid A) : A \subset \{x^0, ..., x^n\}, |A| = k+1\}$ has dimension $\binom{n-k+s}{s}$.*

Proof. By repeated application of (5), all the linear functionals in (2) can be expressed as linear combinations of linear functionals of the form

$$\int_{[A]} f, \qquad A \subset \{x^0, ..., x^n\}, \quad |A| = k+1. \tag{8}$$

So by Theorem 1, the maximum number of linearly independent linear functionals of form (8) is precisely $\dim P_{n-k} = \binom{n-k+s}{s}$. The result then follows from (7). ∎

This result was proved by Dahmen and Micchelli [3], by a very different method, under the slightly stronger assumption that $x^0, ..., x^n$ were k-admissible.

3. INTERPOLATION IN MINIMUM SEMINORM

Suppose Ω is a bounded region in R^s, $s \geqslant 1$. Suppose F is a space of functions on Ω and L a linear operator that maps F to $L^2(\Omega)$. We let Λ denote a set of linear functionals on F and $P \subset F$ an interpolation space with respect to Λ, i.e., for any f in F there is a unique p in P with $\lambda(p) = \lambda(f)$ for all λ in Λ. (In our cases P will be a space of polynomials.) Suppose L is a bijective linear map from a subspace $\mathcal{O}\!\ell \subset F$ to $\mathscr{B} \subset L^2(\Omega)$, where $\mathscr{B}$ (which in our cases will be spanned by B-splines) satisfies the following properties:

(a) If $\lambda(f)=0$ for all λ in Λ, then $\int_\Omega MLf=0$ for all M in $\mathscr{B}$.

(b) If $\int_\Omega MLf=0$ for all M in $\mathscr{B}$, then $\exists p$ in P with $Lp=0$ and $\lambda(f-p)=0$ for all λ in Λ.

Then we have

THEOREM 2. *For any f in F, there is a unique function G in $\mathcal{O}\!\ell + \{p \in P$: $Lp=0\}$ which interpolates f, i.e., $\lambda(G)=\lambda(f)$ for all λ in Λ. Moreover, $\int_\Omega |LG|^2 \leqslant \int_\Omega |Lh|^2$ for all h in F which interpolate f.*

Proof. Let p_0 in P interpolate f. For any h in F which interpolates f, we have $\lambda(h-p_0)=0$ for all λ in Λ. Thus by (a), $L(h-p_0)$ is in

$$\mathscr{B}^\perp := \left\{ g \in L^2(\Omega) \colon \int_\Omega Mg = 0 \text{ for all } M \text{ in } \mathscr{B} \right\},$$

i.e., $Lh \in \mathscr{B}^\perp + Lp_0$.

Now there is a unique element M_0 which minimizes $\int_\Omega |g|^2$ over all g in $\mathscr{B}^\perp + Lp_0$, and it is the unique element of $\mathscr{B}^\perp + Lp_0$ which lies in $\mathscr{B}$. Let H be the unique element of $\mathcal{O}\!\ell$ with $LH = M_0$. Then

$$\int_\Omega |LH|^2 \leqslant \int_\Omega |Lh|^2$$

for all h in F which interpolate f.

Since $M_0 - Lp_0 \in \mathscr{B}^\perp$, we have

$$\int_\Omega ML(H-p_0)=0 \qquad \text{for all } M \text{ in } \mathscr{B}.$$

So by (b), there is a unique p in P with $Lp=0$ and $\lambda(H-p_0-p)=0$ for all λ in Λ, i.e., $\lambda(H-p)=\lambda(p_0)=\lambda(f)$ for all λ in Λ. Then $G=H-p$ satisfies the conditions of the theorem. ∎

Finally, we note that in establishing (a) and (b) it is sufficient to establish (b) only for f in P, i.e., (a) and (b) are equivalent to (a) and

(c) if $p \in P$ and $\int_\Omega MLp=0$ for all M in $\mathscr{B}$, then $Lp=0$.

For suppose (a) and (c) hold, and $\int_\Omega MLf=0$ for all M in $\mathscr{B}$. Let p in P interpolate f, i.e., $\lambda(f-p)=0$ for all λ in Λ. Then by (a), $\int_\Omega ML(f-p)=0$ for all M in $\mathscr{B}$ and hence $\int_\Omega MLp=0$ for all M in $\mathscr{B}$. So by (c), $Lp=0$. Hence (b) is true.

4. Interpolation and B-Splines

Take points $x^0,\ldots,x^n$ in R^s (possibly coincident) and integers k,m with $0 \leqslant k < m \leqslant n$. We denote by $\mathscr{B}_m(x^0,\ldots,x^n)$, or simply $\mathscr{B}_m$, the space spanned by $\{M(\cdot \mid A): A \subset \{x^0,\ldots,x^n\}, |A| = m+1\}$. We shall assume that $x^0,\ldots,x^n$ are q-admissible, $k+1 \leqslant q \leqslant m$. In this case it is well known (see, e.g., [10]) that $\mathscr{B}_m$ comprises piecewise polynomials of degree $m-s$. We denote by Λ_k the set of all linear functionals λ on $C^{q-k-1}(R^s)$ such that for some $A \subset \{x^0,\ldots,x^n\}$, $k+1 \leqslant |A| \leqslant q$, and some α with $|\alpha| = |A| - k - 1$, we have

$$\lambda(f) = \int_{[A]} D^\alpha f.$$

We now return to the setting of Section 3 and let Ω be a bounded region in R^s containing $[x^0,\ldots,x^n]$. We let L denote a linear homogeneous differential operator with constant coefficients of order $m-k$, and put $F = \{f \in C^{q-k-1}(\Omega): Lf \in L^2(\Omega)\}$, where derivatives can be interpreted in the sense of distributions. We put $\Lambda = \Lambda_k \mid F$, $P = P_{n-k} \mid \Omega$, $\mathscr{B} = \mathscr{B}_m \mid \Omega$.

For certain choices of L we shall later construct a space $\mathscr{A}_m$ of functions in R^s such that L is a bijection from $\mathscr{A}_m$ to $\mathscr{B}_m$. We then put $\mathscr{A} = \mathscr{A}_m \mid \Omega$. We note that since elements of $\mathscr{B}_m$ have support in $[x^0,\ldots,x^n]$, the interpolating functions given by Theorem 2 will be independent of the choice of Ω.

By Theorem 1 and Remark 4, we know that P is an interpolation space with respect to Λ. We now show that $\mathscr{B}$ satisfies (a) of Section 3.

LEMMA 1. *If $f \in F$ satisfies $\lambda(f) = 0$ for all λ in Λ, then $\int_\Omega MLf = 0$ for all M in $\mathscr{B}_m$.*

Proof. Take $\{y^0,\ldots,y^m\} \subset \{x^0,\ldots,x^n\}$ and let $M = M(\cdot \mid y^0,\ldots,y^m)$. Then by (7)

$$\int_\Omega MLf = m! \int_{[y^0,\ldots,y^m]} Lf, \qquad \forall f \in C^{m-k}(\Omega).$$

So by repeated application of (5) we see there is a unique element $\mu_{L,M}$ in $\operatorname{Sp}\Lambda$ (the linear span of Λ) such that for all f in $C^{m-k}(\Omega)$:

$$\int_\Omega MLf = \mu_{L,M}(f). \tag{9}$$

Applying this formula to $f(y-\cdot)$ for y in R^s, we have for all f in $C^{m-k}(R^s)$:

$$(-1)^{m-k} M * Lf = \mu_{L,M} * f, \tag{10}$$

where $*$ denotes convolution and $\mu_{L,M}$ can be regarded as a distribution. Now if f is any distribution in R^s, we can find a sequence of functions $\{f_j\}$ in $C^\infty(R^s)$ which converge to f as distributions. Applying (10) to f_j, letting $j \to \infty$, and noting that M and $\mu_{L,M}$ have compact support, we see that (10) holds for any distribution f.

Now take f in F. Then (10) holds and hence so does (9). The result then follows. ∎

We next show $\mathscr{B}$ satisfies (c) of Section 3.

LEMMA 2. *If $p \in P$ and $\int_\Omega MLp = 0$ for all M in $\mathscr{B}$, then $Lp = 0$.*

Proof. Let $\mathscr{B}^*$ denote the space of linear functionals on $\mathscr{B}$. For p in P, we define Φ_p in $\mathscr{B}^*$ by $\Phi_p(M) = \int_\Omega MLp$.

We first show that the map $p \to \Phi_p$ maps P *onto* $\mathscr{B}^*$. For take any Φ in $\mathscr{B}^*$. Then there exists g in $\mathscr{B}$ with $\Phi(M) = \int_\Omega Mg$ for all M in $\mathscr{B}$. Take h in $\mathscr{A}$ with $Lh = g$. By Theorem 1 we can choose p in P such that $\mu(p) = \mu(h)$ for all μ in $\operatorname{sp} \Lambda$. Then for any B-spline M in $\mathscr{B}$

$$\begin{aligned}\Phi(M) = \int_\Omega Mg = \int_\Omega MLh &= \mu_{L,M}(h) \qquad &\text{(by (9))}\\ &= \mu_{L,M}(p) = \int_\Omega MLp \qquad &\text{(by (9))}\\ &= \Phi_p(M).\end{aligned}$$

Thus $\Phi_p = \Phi$ and the map $p \to \Phi_p$ is surjective. Then

$$\begin{aligned}\dim\{p \in P\colon \Phi_p = 0\} &= \dim P_{n-k} - \dim \mathscr{B}^* \\ &= \dim P_{n-k} - \dim P_{n-m} \qquad \text{(by Corollary 1)} \\ &\leqslant \dim\{p \in P_{n-k}\colon Lp = 0\}.\end{aligned}$$

Since $\{p \in P\colon Lp = 0\} \subset \{p \in P\colon \Phi_p = 0\}$, the two spaces must be equal and the lemma is proved. ∎

Thus to establish Theorem 2 for the cases we are considering, it remains only to construct $\mathscr{A}_m \subset C^{q-k-1}(R^s)$ so that L maps $\mathscr{A}_m$ bijectively onto $\mathscr{B}_m$.

Suppose now that L has a fundamental solution K_L, i.e., $LK_L = \delta$, i.e.,

$$(-1)^{m-k} \int_{R^s} K_L(x)\, L\phi(x)\, dx = \phi(0)$$

for all ϕ in $C_0^\infty(R^s) := \{\phi \in C^\infty(R^s)\colon \phi \text{ has compact support}\}$.

Putting $f=K_L$ in (10), we see that for any B-spline M in $\mathscr{B}_m$,

$$M=(-1)^{m-k}\mu_{L,M}*K_L. \tag{11}$$

The map $M\to\mu_{L,M}$ extends to an injective linear map from $\mathscr{B}_m$ to $\Lambda_k^L:=\{\lambda\in\operatorname{sp}\Lambda_k\colon \lambda(p)=0 \text{ for all } p\in P_{n-k} \text{ with } Lp=0\}$. Indeed this map is bijective. To see this, let $P_L=\{p\in P_{n-k}\colon Lp=0\}$ and P_L^* denote the space of linear functionals on P_L. The map from Λ_k to P_L^* given by $\lambda\to\lambda\,|\,P_L$ is surjective by Theorem 1. Its kernel is Λ_k^L and so

$$\begin{aligned}\dim\Lambda_k^L&=\dim\Lambda_k-\dim P_L^*\\&=\dim P_{n-k}-\dim\{p\in P_{n-k}\colon Lp=0\}\\&\leqslant\dim P_{n-m}\\&=\dim\mathscr{B}_m.\end{aligned}$$

Equation (10) tells us that the inverse of the map $M\to\mu_{L,M}$ is the map from Λ_k^L to $\mathscr{B}_m$ given by $\lambda\to(-1)^{m-k}\lambda*K_L$.

It is interesting to examine equation (11) for the simple case $s=1$, $k=0$. In this case $Lf=f^{(m)}$ and K_L is determined, up to a polynomial of degree $\leqslant m-1$, by

$$\begin{aligned}K_L(t)=\frac{1}{(m-1)!}\,t_+^{m-1}:=\frac{1}{(m-1)!}\,t^{m-1},&\qquad \text{if}\quad t\geqslant 0,\\ :=0,&\qquad \text{if}\quad t<0.\end{aligned}$$

By (9), (7), and (1) we have for $M=M(\cdot\,|\,t_0,\ldots,t_m)$,

$$\mu_{L,M}(f)=m![t_0,\ldots,t_m]f$$

and thus (11) becomes the well-known formula

$$M(t\,|\,t_0,\ldots,t_m)=(-1)^m\,m[t_0,\ldots,t_m](t-\cdot)_+^{m-1}.$$

We now consider the case $m-k$ even, $L=\Delta^{(m-k)/2}$, where Δ denotes the Laplacian $\Delta:=\sum_{j=1}^{s}D_j^2$. For $r=1,2,\ldots$, we define

$$\begin{aligned}K_r(x)&=|x|^{2r-s}/c_r, &&\text{if}\quad s\text{ is odd}\quad\text{or}\quad r<\tfrac{1}{2}s,\\&=|x|^{2r-s}\log|x|/c_r, &&\text{if}\quad s\text{ is even}\quad\text{and}\quad r\geqslant\tfrac{1}{2}s,\end{aligned}$$

where

$$c_r=\frac{2^r\pi^{s/2}}{((1/2)s-1)!}\,(r-1)!\,[(2r-s)(2r-s-2)\cdots(4-s)(2-s)],$$

where any factor of zero in the square brackets is to be omitted. It is well known that $\Delta^r K_r = \delta$ (e.g., [13, p. 47]). Thus we may choose $K_L = K_{(m-k)/2}$. Also it is easily checked that $\Delta K_{r+1} = K_r$, $r = 1, 2, \ldots$.

Now take any l with $0 \leqslant l \leqslant s-1$ and $m-l$ even, and put $L' = \Delta^{(m-l)/2}$. Then we define

$$\mathcal{A}_m = \{\lambda * K_{m-k/2-l/2} : \lambda \in \Lambda_l^{L'}\}.$$

Since

$$\begin{aligned} L(\lambda * K_{m-k/2-l/2}) &= \lambda * \Delta^{(m-k)/2} K_{m-k/2-l/2} \\ &= \lambda * K_{(1/2)(m-l)}, \end{aligned}$$

we see L is a bijection from $\mathcal{A}_m$ to $\mathcal{B}_m$. To establish Theorem 2 in this case it remains to show $\mathcal{A}_m \subset C^{q-k-1}(R^s)$.

We note that Λ_l is spanned by linear functionals of the form

$$\lambda(f) = \int_{[A]} D^\alpha f,$$

for $A \subset \{x^0, \ldots, x^n\}$, $|A| \geqslant l+1$, $\mathrm{vol}_s[A] = 0$, $|\alpha| = |A| - l - 1$. For such a λ, suppose A spans a flat of dimension p, i.e., $\mathrm{vol}_p[A] > 0$, $\mathrm{vol}_{p+1}[A] = 0$. Then for any positive integer r, it is not difficult to show that $\lambda * K_r$ satisfies the following properties:

(i) It is C^∞ away from $[A]$.

(ii) It is $C^{2r+l-s+p-|A|}$ everywhere.

(iii) If $|\alpha| = 2r + l - s + p - |A|$, then $D^\alpha(\lambda * K_r)$ restricted to the interior of $[A]$, relative to the flat spanned by $[A]$, is C^∞.

Now for A and p as above, the fact that $x^0, \ldots, x^n$ are q-admissible implies that

$$q + 1 - |A| \geqslant s - p.$$

Applying $q \leqslant m$ and rearranging gives

$$2(m - \tfrac{1}{2}k - \tfrac{1}{2}l) + l - s + p - |A| \geqslant q - k - 1.$$

From (ii) above we then have that $\lambda * K_{m-k/2-l/2}$ is C^{q-k-1} and thus $\mathcal{A}_m \subset C^{q-k-1}(R^s)$.

We next investigate further the properties of elements of $\mathcal{A}_m$. We denote by Γ the union of all sets of the form $[A]$ for subsets A of $\{x^0, \ldots, x^n\}$ with $|A| = s$. It follows from (i)–(iii) above that any function of the form $\lambda * K_r$, $\lambda \in \mathrm{sp}\, \Lambda$, satisfies the following properties:

(i) It is C^∞ on $R^s - \Gamma$.

(ii) Suppose H is an $(s-1)$-dimensional flat spanned by a subset of $\{x^0,\ldots,x^n\}$ and let $A = H \cap \{x^0,\ldots,x^n\}$. Then $\lambda * K_r$ is $C^{2r+l-|A|-1}$ on the interior of $[A]$ relative to Γ.

(iii) With A as in (i) and $|\alpha| = 2r + l - |A| - 1$, then $D^\alpha(\lambda * K_r)$ restricted to the interior of $[A]$ relative to Γ is C^∞.

Putting $r = \frac{1}{2}(m-l)$ and applying (11) gives us the smoothness properties of the B-splines in $\mathscr{B}_m$ (see [3, 6]). Putting $r = m - \frac{1}{2}k - \frac{1}{2}l$ gives us the smoothness properties of $\mathscr{A}_m$. If $n \geqslant 2m-k$, then $\mathscr{B}_{2m-k} \subset \mathscr{A}_m$, and the elements of $\mathscr{A}_m$ satisfy the same smoothness properties as the elements of $\mathscr{B}_{2m-k}$.

Finally, we consider the case $L = D_{y^1} D_{y^2} \cdots D_{y^{m-k}}$, where $y^1,\ldots,y^{m-k}$ are nonzero elements of R^s with $\mathrm{vol}_s[0, y^1,\ldots,y^{m-k}] \neq 0$. Then it is shown in [2, 10] that we can choose K_L to be a piecewise polynomial of degree $m-k-s$. In this case we define

$$\mathscr{A}_m = \{K_L * M : M \in \mathscr{B}_m\}.$$

Since $L(K_L * M) = \delta * M = M$, we see L is a bijection from $\mathscr{A}_m$ to $\mathscr{B}_m$. Also $\mathscr{A}_m$ comprises piecewise polynomials of degree $2m-k-s$.

We shall assume the points $0, y^1,\ldots,y^{m-k}$ are s-admissible (usually called "in general position") and no $(s-1)$-dimensional flat spanned by 0 and a subset of $\{y^1,\ldots,y^{m-k}\}$ is parallel to an $(s-1)$-dimensional flat spanned by a subset of $\{x^0,\ldots,x^n\}$. In this case K_L is $C^{m-k-s-1}$. We have already seen that if

$$r = \max\{|A| : A \subset \{x^0,\ldots,x^n\};\ A \text{ spans a flat of dimension } s-1\},$$

then $\mathscr{B}_m \subset C^{m-r-1}(R^s)$. It follows that $\mathscr{A}_m \subset C^{2m-k-r-1}(R^s)$. Now the fact that $x^0,\ldots,x^n$ are q-admissible implies $r \leqslant q$, and since $q \leqslant m$ we have $2m-k-r-1 \geqslant q-k-1$. Thus $\mathscr{A}_m \subset C^{q-k-1}(R^s)$.

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